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ERGODICITY OF CERTAIN CYLINDER FLOWS

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ABSTRACT

Here we build on the result given in [P1] and extend those in [HL2] to functions which are k times differentiable a.e., $k > 1$. For each k we give a class of irrational numbers S_k such that the skew product extension defined by these functions is ergodic for irrational rotations by these numbers. In the second part of this paper we examine the cohomology of functions over the adding machine transformation, and produce extensions of results from [H1] and [HL3].

1. Introduction

1.1 BACKGROUND. In this paper we will consider cylinder flows defined by the irrational rotation and the von Neumann–Kakutani adding machine transformation on the circle $\mathbf{T}$. These are two types of skew products (or cylinder cascades) which have been studied by various authors. The skew product over the irrational rotation will be dealt with in Section 2, and the skew product over the adding machine will be studied in Section 3. This section will form a basic background of details needed in these later sections.

The conditions for ergodicity, given in Section 2, are on the first k derivatives, $k > 1$, of a continuous function whose integral is zero. This is important when seen in the context of results of L. Baggett [B] and M. Herman [He], when if the function is smooth, then it is a coboundary for certain irrational numbers.

We also note the work of I. Oren [O], who has shown that for rotations by any irrational number α, step functions of the form $\chi_{[0,\beta]} - \beta$ give rise to ergodic

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skew products if and only if β is rational or 1, α and β are linearly independent over the rationals. Also K. Petersen [Pe1] has shown that such step functions define cocycles which are coboundaries if and only if $\beta \in \mathbf{Z}\alpha$. Thus it suffices for us to study continuous functions, and properties of their derivatives which give ergodic skew products for certain irrational numbers.

The results given here extend the work done by P. Hellekalek and G. Larcher in [HL2]. They give results for functions $f : [0,1] \to \mathbf{R}$ which are k times continuously differentiable $k \geq 2$, whose derivatives satisfy:

(i) $f^{(j)}(0) = f^{(j)}(1)$, for $j = 0, \ldots, k-2$,

(ii) $f^{(k-1)}(0) \neq f^{(k-1)}(1)$, and $\int_0^1 f d\mu = 0$.

They give a class of irrational numbers which guarantee ergodicity for this class of functions; for all n these numbers satisfy

$$\|q_n\alpha\| \quad < \quad \frac{1}{q_n^{2k+1}}. \tag{1}$$

Here we weaken the conditions on the final derivatives, we only require that:

(i) $\int f d\mu = 0$ and $f^{(k-2)}$ is continuous,

(ii) $f^{(k-1)}$ is piecewise continuous with zero integral,

(iii) $f^{(k)}$ is Riemann integrable, with non-zero integral.

We prove ergodicity for a wider class of irrational number $\mathcal{S}_k$, whose partial quotients satisfy $\limsup(a_{n+1}/q_n^k) > 0$, that is, there is a positive constant K' such that, for infinitely many n,

$$\|q_n\alpha\| \quad < \quad \frac{K'}{q_n^{k+1}}. \tag{2}$$

These results must also be seen in the light of the work of M. Herman [He, Annexe, p.229], who has demonstrated that C^k functions with zero integral are coboundaries for a class of irrational number which satisfy

$$\|q_n\alpha\| \quad \geq \quad \frac{D}{q_n^r}$$

for all n, where r is any number such that $1 < r < k$. Thus, despite the fact that our functions are close, in some sense, to C^k functions, we get an ergodic skew product for a class of irrational numbers.

Finally, we note that the case $k = 1$ has been covered in the papers [HL1] and [P1]. In [HL1] ergodicity is demonstrated for continuously differentiable functions with a single discontinuity, for all irrationals. In [P1] we give a class of functions which are piecewise absolutely continuous with a derivative which is Riemann integrable with non-zero integral, and show ergodicity for all irrationals. However, if we substitute $k = 1$ in (1) or (2) we obtain a result for a smaller set of irrational numbers than those given for the corresponding functions in [HL1] or [P1] respectively. Thus, it seems that passing from conditions on the first derivative to conditions on the second derivative produces a 'leap' in the requirements on the irrational numbers. This does not appear to occur when passing between any higher derivatives.

The conditions for ergodicity given in Section 3 correspond to those given in Section 2, for $k = 1$. Results on the ergodicity of step functions of zero integral over the adding machine have been given by P. Hellekalek [H2]. For example, in Theorem 2 of [H2] it is shown that step functions of the form $\chi_{[0,\beta)} - \beta$ give rise to ergodic skew products if and only if β is irrational, or strictly non-q-adic. Theorem 3 in the same paper, [H2], gives us results for more general step functions. Also from [H1], we know that such step functions define cocycles which are coboundaries if and only if β is q-adic.

Firstly, for the basic $k = 1$ case, we demonstrate ergodicity of the skew product for all adding machine transformations. This extends the work of P. Hellekalek and G. Larcher [HL3] who use a Lipschitz derivative condition for functions with a single, non-zero jump on a particular type of adding machine. We then observe that using the methods of Section 2 does not give us an extension of this result for $k > 1$.

In the latter part of this section we give a class of functions, namely those with zero integral, and have a derivative of bounded variation. We show that these functions define cocycles which are always coboundaries for a certain class of adding machine transformation. This is analogous to the result by M. Herman [He], who has shown that this same class of functions are also coboundaries for almost all irrational numbers.

This result, together with the class of functions given in the first part of this section, emphasize that certain adding machine transformations have features in common with rotations by irrational numbers with bounded partial quotients. We also note that the functions given in the first part are close, in some sense, to those given in the second; however, the discontinuities which give a non-zero integral for the derivative make all the difference for the cohomology of the cocycle.

1.2 COCYCLES, ESSENTIAL VALUES, SKEW PRODUCTS. Firstly some definitions and notation in common use throughout this paper. All of the proofs of the results given here are to be found in [KS].

1.2.1 Definitions: Let $(\mathbf{T}, \Omega, \mu)$ denote the one-dimensional torus with standard Borel field and Lebesgue measure. If T is an ergodic automorphism of $\mathbf{T}$, then this defines a $\mathbf{Z}$-action $n \mapsto T^n$, $n \in \mathbf{Z}$ on $(\mathbf{T}, \Omega, \mu)$. A **real-valued additive cocycle for** T is a Borel map $a : \mathbf{Z} \times \mathbf{T} \to \mathbf{R}$ satisfying the cocycle relation:

$$a(n+m, x) = a(n, T^m x) + a(m, x) \tag{3}$$

for μ-a.e. $x \in \mathbf{T}$, and all $n, m \in \mathbf{Z}$. Any such cocycle is uniquely determined by the function $f(\cdot) = a(1, \cdot)$. Thus we have

$$a(n,x) = \begin{cases} \sum_{k=0}^{n-1} f \circ T^k(x) & \text{for } n \geq 1, \\ 0 & \text{for } n = 0, \\ -a(-n, x) & \text{for } n \leq -1. \end{cases}$$

Conversely, if $f : \mathbf{T} \to \mathbf{R}$ is any Borel map, the above formulae define a real-valued cocycle for T. We write $f_n(x)$ for $a(n, x)$, as defined above, to indicate the relationship between the cocycle and its determining function.

The cocycle a is a **coboundary** if and only if there is a Borel map $g : \mathbf{T} \to \mathbf{R}$ such that for μ-a.e. $x \in \mathbf{T}$, $a(1, x) = f(x) = g(Tx) - g(x)$.

Let $\overline{\mathbf{R}} = \mathbf{R} \bigcup \{\infty\}$ be the one-point compactification of $\mathbf{R}$. An element $y \in \overline{\mathbf{R}}$ is called an **essential value of** a if, for every neighbourhood $N(y)$ of y in $\overline{\mathbf{R}}$ and for every $B \in \Omega$ with $\mu(B) > 0$, we have an $n \in \mathbf{Z}$ such that

$$\mu\left(B \bigcap T^{-n} B \bigcap \{ x : a(n, x) \in N(y) \}\right) > 0.$$

The set of all essential values of the cocycle a is denoted by $\overline{E}(a)$ and we put $E(a) = \overline{E}(a) \bigcap \mathbf{R}$. ∎

The following are standard results from [KS]:

1.2.2 LEMMA: *$E(a)$ is a closed (additive) subgroup of* $\mathbf{R}$.

1.2.3 LEMMA: *Let T be an ergodic automorphism of $(\mathbf{T}, \Omega, \mu)$ and let $a : \mathbf{Z}\times\mathbf{T} \to \mathbf{R}$ be a cocycle for T. Suppose $\mathcal{K} \subset \mathbf{R}$ is a compact set with $\mathcal{K} \bigcap E(a) = \emptyset$, then for every $C \in \Omega$ with $\mu(C) > 0$ there is a Borel set $B \subset C$ with $\mu(B) > 0$ such that for all $n \in \mathbf{Z}$, $B \bigcap T^{-n}B \bigcap \{x : a(n,x) \in \mathcal{K}\} = \emptyset$.*

1.2.4 Definition: Let $(\mathbf{R}, \mathcal{B}, \lambda)$ denote the standard Lebesgue structure on $\mathbf{R}$. We form a new measure space $(\mathbf{T} \times \mathbf{R}, \Omega \times \mathcal{B}, \mu \times \lambda)$ with product structure, suppose that $a : \mathbf{Z} \times \mathbf{T} \to \mathbf{R}$ is a cocycle for the ergodic automorphism T on $\mathbf{T}$, and define a new action T_a of $\mathbf{Z}$ on $\mathbf{T} \times \mathbf{R}$ by $T_a^n(\alpha, x) = (T^n\alpha, x + a(n, \alpha))$.

The $\mathbf{Z}$-action T_a on $\mathbf{T} \times \mathbf{R}$ is called the **skew product of T with $\mathbf{R}$** . ∎

The main result we need comes from [KS], and relates the ergodicity of T_a to the properties of the cocycle:

1.2.5. THEOREM: *T_a is ergodic if and only if $E(a) = \mathbf{R}$.*

1.3 THE IRRATIONAL ROTATION. When there is no ambiguity we shall think of arbitrary real numbers as elements of $\mathbf{T}$, by identifying them with their congruence class mod 1. We shall also employ the following notation:

For $x \in \mathbf{R}$ we denote by $\|x\|$ the distance of x to the nearest integer:

$$\|x\| = \min\{\, |\, j - x\,| : j \in \mathbf{Z}\,\}.$$

For $\alpha \in \mathbf{T}$, we denote by $[a_1, a_2, \ldots]$ the **continued fraction expansion** of α. The a_k are called the **partial quotients** of α.

We define

$$\frac{p_k}{q_k} = [a_1, \ldots, a_k] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_k}}}}$$

in lowest terms, for $k \geq 1$. These fractions are called the **convergents** of α and the q_k are called the **partial quotient denominators.**

These partial quotient denominators tell us which multiples of α are closest to integers; in particular we have the following:

1.3.1 LEMMA:

(i) *For all n we have that $\frac{1}{2} < q_n \cdot \|q_{n-1}\alpha\| < 1$,*

(ii) *for all n and all $p > 0$ with $q_n^p < q_{n+1}$ we have* $\|q_n^p\alpha\| \le q_n^{p-1} \cdot \|q_n\alpha\|$.

Proof: For the first equation, see [HW] or [Ca]. For the second, we note that if j is the integer for which $| j - q_n\alpha |$ is minimized, then $\|q_n^p\alpha\| \le | q_n^{p-1}.j - q_n^p\alpha |$ and so by the triangle inequality,

$$\|q_n^p\alpha\| \le | j - q_n\alpha | + \cdots + | j - q_n\alpha | = q_n^{p-1} \cdot \|q_n\alpha\|,$$

which proves the result. ∎

1.4 THE ADDING MACHINE. As in [H1], we shall consider a generalization of the von Neumann–Kakutani adding machine transformation on $\mathbf{T}$; the following definitions and notation are taken from those given in [H2].

Let $\mathbf{q} = (q_i)_{i\ge1}$ be a sequence of integers q_i, with $q_i \ge 2$, for all i. If $\mathbf{A}$ $(\mathbf{q})$ denotes the compact Abelian group of $\mathbf{q}$-adic integers, then the transformation $\mathbf{z} \mapsto \mathbf{z} + \mathbf{1}$ on $\mathbf{A}(\mathbf{q})$, where $\mathbf{1} = (1, 0, \ldots)$, is uniquely ergodic with respect to normalised Haar measure on $\mathbf{A}$ $(\mathbf{q})$ (see [HR]).

Consider next the circle, $\mathbf{T}$ with Haar measure μ. Define the sequence of integers $p(k)$, $k \ge 0$ by

$$p(k) = \begin{cases} 1 & \text{for } k = 0, \\ q_1 \ldots q_k & \text{for } k = 1, 2, \cdots. \end{cases}$$

If $\mathbf{z} = \Sigma_{i=0}^{\infty} z_i p(i)$ with $z_i \in \{0, 1, \ldots, q_{i+1} - 1\}$ is an element of $\mathbf{A}(\mathbf{q})$, then

$$\Phi(\mathbf{z}) = \sum_{i=0}^{\infty} \frac{z_i}{p(i+1)} \mod 1$$

belongs to $\mathbf{T}$. The map $\Phi : \mathbf{A}(\mathbf{q}) \to \mathbf{T}$ is measure-preserving and injective on $\mathbf{A}(\mathbf{q})$ except on a subset of Haar measure zero.

The $\mathbf{q}$-adic representation of an element x of $\mathbf{T}$,

$$x = \sum_{i=0}^{\infty} \frac{x_i}{p(i+1)} \quad \text{with } x_i \in \{0, 1, \ldots, q_{i+1} - 1\},$$

is unique under the condition $x_i \ne q_{i+1} - 1$ for infinitely many i. The uniqueness condition for the representation ensures that the following transformation $T : \mathbf{T} \to \mathbf{T}$ is well defined:

$$Tx := \Phi(\mathbf{z} + 1) \quad \text{where } \mathbf{z} = \mathbf{z}(x) = \sum_{i=0}^{\infty} x_i p(i).$$

T is ergodic with respect to μ and $T \circ \Phi(\mathbf{z}) = \Phi(\mathbf{z} + \mathbf{1})$ for almost all $\mathbf{z} \in \mathbf{A}(\mathbf{q})$. T may be called a (generalized) **von Neumann–Kakutani adding machine transformation**, with associated integers $p(k)$, $k \ge 1$.

2. The Irrational Rotation

2.1 INTRODUCTION. In this section we study skew product extensions of irrational rotations on $\mathbf{T}$. For each $k > 1$ we define a class of functions which have k derivatives a.e. satisfying various conditions. We study this class of functions and give, for each k, a class of irrational number $\mathcal{S}_k$ and show that skew products over rotations by numbers in this class are ergodic.

In the subsection below, firstly we discuss the orbits of the irrational rotation and define the classes of functions and irrational numbers that we shall be working with. Then we study the cocycles defined by functions in our class over rotations by these irrational numbers. Finally we demonstrate that the only possibility for their group of essential values (or asymptotic range) is $\mathbf{R}$, which guarantees ergodicity of the skew product (see 1.2.5).

2.2 CONDITIONS FOR ERGODICITY.

2.2.1 Convention: Throughout this section we shall suppose that f is k-times differentiable a.e. $k > 1$, with derivatives satisfying the following properties:
(i) $\int f d\mu = 0$ and $f^{(k-2)}$ is continuous,
(ii) $f^{(k-1)}$ is piecewise continuous with zero integral,
(iii) $f^{(k)}$ is Riemann integrable with $\int f^{(k)} d\mu \neq 0$.

For the rest of this subsection we assume without loss of generality that $\int f^{(k)} d\mu > 0$, and that the irrational number $\alpha < \frac{1}{2}$. ∎

For small sub-intervals of $\mathbf{T}$ we shall consider the order inherited from $\mathbf{R}$, and use the words left and right accordingly.

The following result may be found in part 1, section 4 of [K].

2.2.2 PROPOSITION: *Let $P_n(\alpha)$ be the set of right half-open intervals of $\mathbf{T}$ defined by the points $\{-j\alpha\}$ for $j = 0, \ldots, q_n - 1$. Then for all n, each interval of $P_n(\alpha)$ has length $\|q_{n-1}\alpha\|$ or $\|q_n\alpha\| + \|q_{n-1}\alpha\|$. For all n, the map T sends each interval of $P_n(\alpha)$ onto another, with the following exceptions: for n even*
(i) *the interval $[0, -q_{n-1}\alpha)$ is placed inside $[-(q_n - 1)\alpha, -(q_{n-1} - 1)\alpha)$,*
(ii) *the interval $[-(q_n - q_{n-1})\alpha, 0)$ overflows $[-(q_n - q_{n-1} - 1)\alpha, -(q_n - 1)\alpha)$.*

For odd n the intervals are the same but the end-points are swapped.

For $j = 0, \ldots, k-1$ we denote the variation of $f^{(j)}$ by $c(j)$, then the Denjoy–Koksma inequality gives us that

$$| f_{q_n}^{(j)}(x) | \leq c(j), \tag{4}$$

for each $j = 0, \dots, k-1$, all $n \geq 0$ and all $x \in \mathbf{T}$.

Let $\omega_1, \dots, \omega_N$ be the discontinuities of $f^{(k-1)}$. Arrange these discontinuities in increasing order: $0 \leq \omega_1 \leq \cdots \leq \omega_N < 1$. We may assume that $\omega_1 = 0$, since the rotation of the domain of f necessary to bring this about may be performed initially, and does not affect the ergodicity in question.

Suppose that at ω_r, $f^{(k-1)}$ jumps by $d_r = f^{(k-1)}(\omega_r^+) - f^{(k-1)}(\omega_r^-)$ for $r = 1, \dots, N$, where $f^{(k-1)}(\omega_r^+)$, $f^{(k-1)}(\omega_r^-)$ are the limits of $f^{(k-1)}(x)$ as x approaches ω_r from the left, right respectively. Since $f^{(k-1)}$ is piecewise continuous we have the following relationship:

2.2.3 LEMMA: $\int f^{(k)} d\mu \; = \; - \sum_{r=1}^{N} d_r$.

Hence the condition for $\int f^{(k)} d\mu \neq 0$ is equivalent to $\sum d_r \neq 0$. From [P1, Lemma 2.6] we also have the following:

2.2.4 PROPOSITION: *For all n, the cocycle $f_{q_n}^{(k-1)}$ has discontinuities at $\{w_r - j\alpha\}$ for $0 \leq j < q_n$ and $r = 1, \dots, N$, with jumps of size d_r at these points. Also, each partition interval Q_n of $P_n(\alpha)$, contains at most $2N-2$ discontinuities of $f_{q_n}^{(k-1)}$.*

2.2.5 Definition: For any positive integer $k > 1$, define the subset S_k of irrational numbers whose partial quotients $\{a_n\}$, $n \geq 1$ satisfy the following condition:

$$\limsup \frac{a_{n+1}}{q_n^k} \; > \; 0.$$

For any $\alpha \in S_k$, suppose that $\limsup(a_{n+1}/q_n^k) = S > 0$. so there exists a subsequence of integers $\{n_j\}_{j \geq 1}$ and an integer J, with $a_{n_j+1} \geq K q_{n_j}^k$ for some $0 < K < S$ and all $j > J$. Since this is the subsequence we shall be considering, for convenience we drop the subscripts and assume that there is a positive constant K and an integer N' such that

$$(5) \qquad a_{n+1} \geq K q_n^k \quad \text{for } n \geq N'.$$

For the rest of this section we shall assume that $\alpha \in S_k$.

2.2.6 LEMMA: *There is a positive constant Y, such that for $n > N'$ and $j = 0, \dots, k-1$, $\mid f_{q_n}^{(j)}(x) \mid < \frac{Y}{q_n^{k-1-j}}$ for all $x \in \mathbf{T}$.*

Proof: The result is certainly true for $j = k-1$ from (4) above, for all n, with $Y = Y(k-1) = c(k-1)$. We suppose that the formula holds for some

$1 \le j \le k-1$ with $Y = Y(j)$ and demonstrate the inequality for $j-1$, $n > N'$ and some $Y = Y(j-1)$. Then, if we take Y to be the maximum of all these $Y(j)$ for $j = 0, \dots, k-1$ this will prove our result.

We proceed in three steps: Firstly we show that there is a positive constant E such that for $n > N'$, and all intervals $Q_n \in P_n(\alpha)$,

$$\left| \int_{Q_n} f_{q_n}^{(j-1)} d\mu \right| < \frac{E}{q_n^{k+1}}. \tag{6}$$

From Proposition 2.2.2 and the cocycle relation (3), the integral in question is the integral of $f^{(j-1)}$ over $q_n - 1$ iterates of Q_n under T. Suppose that $\mu(Q_n) = \|q_{n-1}\alpha\|$, then the integral is equal to the integral of $f^{(j-1)}$ over all of $\mathbf{T}$, except for q_{n-1} intervals of size $\|q_n\alpha\|$ which are iterates under T. Hence

$$\int_{Q_n} f_{q_n}^{(j-1)} d\mu = \int_{Q_n} \sum_{p=0}^{q_{n-1}-1} f^{(j-1)}(T^p x) d\mu(x)$$
$$= \int_{\bigcup_{p=0}^{q_{n-1}-1} T^p Q_n} f^{(j-1)} d\mu = \int_{\mathbf{T}} f^{(j-1)} d\mu - \int_I f_{q_{n-1}}^{(j-1)} d\mu,$$

where I is the interval of size $\|q_n\alpha\|$, whose q_{n-1} iterates are missed out. Thus we have

$$\left| \int_{Q_n} f_{q_n}^{(j-1)} d\mu \right| = \left| \int_I f_{q_{n-1}}^{(j-1)} d\mu \right|,$$

since $f^{(j-1)}$ has zero integral over $\mathbf{T}$. Applying (4), we get

$$\left| \int_{Q_n} f_{q_n}^{(j-1)} d\mu \right| \le c(j-1)\|q_n\alpha\| < \frac{c(j-1)}{q_n a_{n+1}} < \frac{c(j-1)}{K q_n^{k+1}},$$

for $n > N'$, since $\alpha \in \mathcal{S}_k$.

If $\mu(Q_n) = \|q_{n-1}\alpha\| + \|q_n\alpha\|$, then we split this interval into two chunks, one I_1 of size $\|q_{n-1}\alpha\|$, the other I_2 of size $\|q_n\alpha\|$. For I_1 we get the above estimate for the integral of $f_{q_n}^{(j-1)}$ over it. For I_2 we have that for $n > N'$

$$\left| \int_{I_2} f_{q_n}^{(j-1)} d\mu \right| \le c(j-1)\|q_n\alpha\| < \frac{c(j-1)}{K q_n^{k+1}},$$

also by (4). Therefore, if we choose $E = 2c(j-1)/K$, then we have established (6) for all intervals $Q_n \in P_n(\alpha)$, and all $n > N'$.

For the second step we demonstrate the following: There is a positive constant G such that, for $n > N'$, in each interval $Q_n \in P_n(\alpha)$ there is an $x \in Q_n$ with

$$\left|f_{q_n}^{(j-1)}(x)\right| < \frac{G}{q_n^k}. \tag{7}$$

Suppose not, then for some $Q_n \in P_n(\alpha)$ and any $G > 0$ we have that

$$| f_{q_n}^{(j-1)}(x) | \geq \frac{G}{q_n^k} \quad \text{for all} \quad x \in Q_n \quad \text{and some} \quad n > N'.$$

In particular this holds for $G = 4E$. Supposing, without loss of generality, that $f_{q_n}^{(j-1)}$ is strictly positive on Q_n, we must have that

$$\left| \int_{Q_n} f_{q_n}^{(j-1)} d\mu \right| > \frac{G}{q_n^k} \cdot \mu(Q_n) \geq \frac{G}{q_n^k} \cdot \|q_{n-1}\alpha\|,$$

from Proposition 2.2.2. However, from (6) we have an upper bound for this integral, and since $n > N'$, we get

$$\frac{G}{q_n^k} \cdot \|q_{n-1}\alpha\| < \frac{E}{q_n^{k+1}}.$$

Hence we have that

$$q_n \cdot \|q_{n-1}\alpha\| < \frac{E}{G} = \frac{1}{4},$$

which contradicts 1.3.1(i). This establishes (7).

Finally we show the following: There is a constant $Y(j-1)$ such that for $n > N'$ and any interval Q_n of $P_n(\alpha)$, for all $x \in Q_n$, we have

$$\left|f_{q_n}^{(j-1)}(x)\right| < \frac{Y(j-1)}{q_n^{k-j}}.$$

Since $f^{(j-1)}$ is absolutely continuous and the formula is true for j we have, for all $a, b \in \mathbf{T}$, that

$$\left|f_{q_n}^{(j-1)}(b) - f_{q_n}^{(j-1)}(a)\right| = \left| \int_a^b f_{q_n}^{(j)} d\mu \right| < \frac{Y(j)}{q_n^{k-1-j}} \cdot |b-a|.$$

Since $a, b \in Q_n$, we have $|\, b - a \,| < 2\|q_{n-1}\alpha\|$ and, using 1.3.1(i), this becomes

$$\left| f_{q_n}^{(j-1)}(b) - f_{q_n}^{(j-1)}(a) \right| < \frac{Y(j)}{q_n^{k-1-j}} \cdot 2\,\|q_{n-1}\alpha\| < \frac{2Y(j)}{q_n^{k-j}}.$$

From (7) we may choose a to be a point for which $\left| f_{q_n}^{(j-1)}(a) \right| < G/q_n^k$, hence we have for $n > N'$ and all $x \in Q_n$ that

$$\left| f_{q_n}^{(j-1)}(x) \right| < \frac{G}{q_n^k} + \frac{2Y(j)}{q_n^{k-j}} \le \frac{2Y(j) + G}{q_n^{k-j}}.$$

This proves the result for $j-1$, $n > N'$ with $Y(j-1) = 2Y(j) + G$, and the rest follows by induction. ∎

Since $f^{(k)}$ is Riemann integrable we may use Weyl's Theorem (see [Pe2, p.50]) to obtain the following, using the proof from [P1, Lemma 2.3].

2.2.7 LEMMA: *There are positive constants $K_1(k)$, $K_2(k)$ and a positive integer Z such that for $n > Z$ we have*

$$n.K_1(k) < f_n^{(k)}(x) < n.K_2(k)$$

for all $x \in \mathbf{T}$.

2.2.8 PROPOSITION: *Suppose that g is a function which defines the cocycle g_{q_n}, for which there are positive constants R, L_1, L_2 and an integer N_1 such that for $n > N_1$, all partition intervals Q_n of $P_n(\alpha)$ contain a subinterval $J_n = [a_n, b_n)$ which satisfies*

$$\mu(J_n) \ge \frac{1}{R}\mu(Q_n),$$

on which g_{q_n} is absolutely continuous and

$$q_n.L_1 < g'_{q_n}(x) < q_n.L_2 \quad \text{or} \quad -q_n.L_2 < g'_{q_n}(x) < -q_n.L_1$$

for $n > N_1$ and all $x \in J_n$. Then there are constants H and F such that for $n > N_1$,

(i) *g_{q_n} moves through a height greater than H on J_n,*

(ii) *for any interval $I_n = [x, y] \subseteq g_{q_n}(J_n)$ we have that*

$$\frac{\mu\left(g_{q_n}^{-1}(I_n) \bigcap J_n\right)}{\mu(Q_n)} > F\,|\, y - x \,|\,.$$

Proof: Choose $H = L_1/4R$, and suppose that g_{q_n} is strictly increasing for $n > N_1$, then the height h moved through by the cocycle on J_n is given by

$$h = g_{q_n}(b_n) - g_{q_n}(a_n) = \int_{a_n}^{b_n} g'_{q_n} d\mu.$$

By hypothesis, applying Proposition 2.2.2 and Lemma 1.3.1(i), we have that for $n > N_1$

$$h > \frac{L_1}{R} \cdot q_n \cdot \mu(Q_n) > \frac{L_1}{2R}.$$

If g_{q_n} is strictly decreasing we obtain, also for $n > N_1$, a reversed inequality, but with a minus sign; this proves (i).

For (ii), define $m = g_{q_n}^{-1}(x) \bigcap J_n$ and $p = g_{q_n}^{-1}(y) \bigcap J_n$; then, supposing that g_{q_n} is strictly increasing, for $n > N_1$ we have $g_{q_n}^{-1}(I_n) \bigcap J_n = [m, p)$, and

$$y - x = \int_m^p g'_{q_n} d\mu.$$

By definition of g'_{q_n} we have, for $n > N_1$, that

$$p - m > \frac{y - x}{q_n . L_2}.$$

Hence we obtain

$$\frac{p - m}{\mu(Q_n)} > \frac{y - x}{2L_2 . q_n \|q_{n-1}\alpha\|},$$

and so, using Lemma 1.3.1(i),

$$\frac{\mu\left(g_{q_n}^{-1}(I_n) \bigcap J_n\right)}{\mu(Q_n)} > \frac{y - x}{2L_2}.$$

If g_{q_n} is strictly decreasing, we get the same inequality for $n > N_1$, so putting $F = 1/2L_2$ completes the proof of (ii). ∎

To simplify the statement and proof of the following lemma we shall introduce some extra notation: For the cocycle $f_{q_n^r}^{(s)}$ we shall write $\phi_r^{(s)}$, where $0 \leq r, s \leq k$.

2.2.9 Lemma: *For $j = 0, \ldots, k-1$ there exist strictly positive constants $H(j)$, $F(j)$, $L(j)$ and an integer $N_1(j)$ with the following properties: Within each interval Q_n of $P_n(\alpha)$ there is a subinterval $J_n(j)$, with*

$$\mu(J_n(j)) > \frac{1}{L(j)} \mu(Q_n)$$

such that for $n > N_1(j)$,
(i) $\phi^{(j)}_{k-j}$ *moves through a height greater than $H(j)$ on $J_n(j)$,*
(ii) *for any interval $I_n(j) = [x, y]$ in $\phi^{(j)}_{k-j}(J_n(j))$ we have that*

$$\frac{\mu\left(\left(\phi^{(j)}_{k-j}\right)^{-1}(I_n(j))\bigcap J_n(j)\right)}{\mu(Q_n)} > F(j)\,|\,y-x\,|\,.$$

Proof: The result is certainly true for $j = k-1$ by using Lemma 2.2.7, Proposition 2.2.4, and applying Proposition 2.2.8 to $g = \phi_0^{(k-1)}$ with $R = 1/(2N-2)$, $L_1 = K_1(k)$, $L_2 = K_2(k)$ and N_1 such that $q_{N_1} > Z$. We shall assume that the result holds for $1 \le j \le k-1$, and demonstrate the result for $j-1$.

From (i), since the result holds for j, for $n > N_1(j)$, $\phi^{(j)}_{k-j}$ moves through at least $H(j)$ on every $J_n(j) \subseteq Q_n$ where

$$\mu\left(J_n(j)\right) \ge \frac{1}{L(j)}\mu(Q_n).$$

Hence there is some $x \in J_n(j)$ with $\left|\phi^{(j)}_{k-j}(x)\right| > \frac{1}{4}H(j)$, so without loss of generality we may suppose that $\phi^{(j)}_{k-j}(x) > \frac{1}{4}H(j)$. By hypothesis $\phi^{(j)}_{k-j}$ is continuous on $J_n(j)$ so, for $n > N_1(j)$, we may apply (ii) above to the interval

$$I_n(j) \;=\; \left[\frac{1}{4}H(j), Y\right]\bigcap \phi^{(j)}_{k-j}\left(J_n(j)\right),$$

to give us $S_n(j) \;=\; \left(\phi^{(j)}_{k-j}\right)^{-1} I_n(j)\bigcap J_n(j)$ satisfying

$$\mu\left(S_n(j)\right) > \frac{1}{L'(j)}\mu(Q_n),$$

on which

$$t_1(j) \;<\; \phi^{(j)}_{k-j}(x) \;<\; t_2(j) \tag{8}$$

for some strictly positive constants $t_1(j)$, $t_2(j)$ and $L'(j)$.

From the cocycle relation (3), for all n and all $x \in \mathbf{T}$, we have that

$$\phi^{(j)}_{k-j+1}(x) \;=\; \phi^{(j)}_{k-j}(x) + \phi^{(j)}_{k-j}(W_{k-j}x) \;+\; \cdots \;+\; \phi^{(j)}_{k-j}(W_{k-j}^{q_n-1}x),$$

where $W_{k-j} = T^{q_n^{k-j}}$. Hence, in order to estimate bounds for $\phi^{(j)}_{k-j+1}(x)$ we must look at the values of $\phi^{(j)}_{k-j}$ for $q_n - 1$ iterates of x by W_{k-j}. Since $\alpha \in S_k$ we have, for $n > N'$ and $1 \le M \le q_n - 1$, that

$$|\,W^M_{k-j}x - x\,| = \;M\,.\,\|q_n^{k-j}\alpha\|$$

which, by Lemma 1.3.1(ii) and (5), becomes

$$\leq M.q_n^{k-j-1}\|q_n\alpha\| < M.q_n^{k-j-1}\cdot\frac{1}{Kq_n^{k+1}} = \frac{M}{Kq_n^{j+2}}.$$

Hence for $n > N_1(j-1) = \max\{N', N_1(j)\}$, all points in $S_n(j)$, except for a subinterval I at one end, of size $\mu(I) < 1/Kq_n^{j+1}$, return to $S_n(j)$ at least $q_n - 1$ times under W_{k-j}. Thus from (8), for $x \in V_n(j) = S_n(j)\backslash I$ we have, for $n > N_1(j-1)$, that

$$q_n.t_1(j) < \phi_{k-j+1}^{(j)}(x) < q_n.t_2(j).$$

Applying Proposition 2.2.8 for $g = \phi_{k-j}^{(j-1)}$, which is absolutely continuous on J_n, since by hypothesis $\phi_{k-j}^{(j)}$ is continuous on $J_n(j) \supset J_n$, and interval $J_n = V_n(j)$ gives us the result for $j-1$. The result follows by induction. ∎

2.2.10 LEMMA: *For any $A \subseteq \mathbf{T}$ with $\mu(A) > 0$, and every $\epsilon > 0$, there is an $A_0 \subseteq A$ with $\mu(A\backslash A_0) < \epsilon$ and an infinite sequence $\{n_i\}_{i\geq 1}$ of integers such that $T^{q_{n_i}^k}x \in A$ for all i, and all $x \in A_0$.*

Proof: For any Borel set B of positive measure the map $x \mapsto \mu(B\bigcap(B-x))$ is continuous at 0, where $B - x$ denotes the set B translated by the element $-x \in \mathbf{T}$ (cf. [R, Theorem 1.1.5]).

So, given $\epsilon > 0$, there is a $\delta > 0$ such that $|\ \mu(B) - \mu(B\bigcap(B-x))\ | < \epsilon$ for $|\ x\ | < \delta$. From Lemma 1.3.1(ii) and (5) we have that

$$\|q_n^k\alpha\| \leq q_n^{k-1}\|q_n\alpha\| \leq \frac{a_{n+1}}{Kq_n}\|q_n\alpha\|.$$

For all n we have that $\|q_n\alpha\| < 1/a_{n+1}q_n$, and so

$$\|q_n^k\alpha\| < \frac{1}{Kq_n^2} \to 0 \qquad \text{as } n \to \infty.$$

Hence given $\delta > 0$ we may choose $K'' > 0$ such that, for $n > K''$, we have $\|q_n^k\alpha\| < \delta$, and hence

$$\left|\mu(B) - \mu\left(B\bigcap(B-x)\right)\right| = \left|\mu(B) - \mu\left(B\bigcap T^{-q_n^k}B\right)\right| < \epsilon.$$

If we apply the above to our set A, given $\epsilon > 0$ we can find an n_1 such that

$$\mu\left(A\bigcap T^{-q_{n_1}^k}A\right) > \mu(A) - \frac{\epsilon}{2}. \tag{9}$$

Now we apply the above argument to the set $A \bigcap T^{-q^k_{n_1}} A$ and obtain an n_2 with

$$(10) \quad \mu\left(A \bigcap T^{-q^k_{n_1}} A \bigcap T^{-q^k_{n_2}} A \bigcap T^{-q^k_{n_1}-q^k_{n_2}} A\right) > \mu\left(A \bigcap T^{-q^k_{n_1}} A\right) - \frac{\epsilon}{4}.$$

Then by monotonicity, (9) and (10) we have that

$$\mu\left(A \bigcap T^{-q^k_{n_1}} A \bigcap T^{-q^k_{n_2}} A\right) \geq \mu\left(A \bigcap T^{-q^k_{n_1}} A \bigcap T^{-q^k_{n_2}} A \bigcap T^{-q^k_{n_1}-q^k_{n_2}} A\right)$$
$$> \mu\left(A \bigcap T^{-q^k_{n_1}} A\right) - \frac{\epsilon}{4} > \mu(A) - \frac{3}{4}\epsilon.$$

Inductively, we get for $i \geq 1$

$$\mu\left(A \bigcap T^{-q^k_{n_1}} A \bigcap \cdots \bigcap T^{-q^k_{n_i}} A\right) > \mu(A) - \left(\frac{2^i - 1}{2^i}\right)\epsilon.$$

Letting $i \to \infty$ and defining $q_{n_0} = 0$, we have that

$$\mu\left(\bigcap_{i=0}^{\infty} T^{-q^k_{n_i}} A\right) > \mu(A) - \epsilon.$$

The set $A_0 = \bigcap_{i=0}^{\infty} T^{-q^k_{n_i}} A$ has the required properties ∎

From [P1, Lemma 2.9] we have the following:

2.2.11 Lemma: *Given $A \subseteq \mathbf{T}$ with $\mu(A) > 0$, and $\epsilon' > 0$, there is an $A_0 \subseteq A$ with $\mu(A_0) > 0$ and $\mu(A \backslash A_0) < \epsilon'$, with the following properties: Given any $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in A_0$, and all intervals $I(x)$ containing x of length $\mu(I(x)) < \delta$, we have that*

$$\frac{\mu\left(I(x) \bigcap A\right)}{\mu\left(I(x)\right)} > 1 - \epsilon.$$

2.2.12 Lemma: *$E(f) \neq \lambda \mathbf{Z}$ for any $\lambda \geq 0$.*

Proof: Suppose that $E(f) = \lambda\mathbf{Z}$ where $\lambda \geq 0$. By the cocycle relation (3) and Lemma 2.2.6 we have that $| f_{q^k_n}(x) | < Y$ for all $n > N'$, and all $x \in \mathbf{T}$. Let v be the greatest integer such that $v\lambda < Y$, with $v = 0$ if $\lambda = 0$. We choose

$$0 < \epsilon < \frac{H(0)}{4(2v+1)}$$

where $H(0)$ is obtained in Lemma 2.2.9(i). Consider the compact set

$$\mathcal{K} = (-\epsilon, Y] \setminus \bigcup_{i=0}^{v} (i\lambda - \epsilon, i\lambda + \epsilon).$$

This is the interval $(-\epsilon, Y]$ without intervals of width 2ϵ about each $i\lambda \in E(f)$, for $i = 0, \ldots, v$. Clearly $\mathcal{K} \bigcap E(f) = \emptyset$, so applying Lemma 1.2.3 gives us a Borel set $B \in \Omega$ with $\mu(B) > 0$ such that for all $m \in \mathbf{Z}$, we have $B \bigcap T^{-m} B \bigcap \{y : | f_m(y) | \in \mathcal{K}\} = \emptyset$.

From Lemma 2.2.9(i), for all $n > N_1(0)$, every interval Q_n in $P_n(\alpha)$ contains a subinterval $J_n(0)$ on which $| f_{q_n^k} | = | \phi_k |$ moves continuously through a height greater than $\frac{1}{2} H(0)$. We claim that for n sufficiently large, and for every interval Q_n, the ratio of the Lebesgue measure of Q_n to that of the set $Q_n \bigcap \{y : | f_{q_n^k}(y) | \in \mathcal{K}\}$ is greater than a fixed positive number.

Applying 2.2.9(ii) to the interval(s) comprising $\mathcal{K}$ of total length at least $\frac{1}{2} H(0) - (2v+1)\epsilon > \frac{1}{4} H(0)$, through which the graph of $| f_{q_n^k} |$ must pass, gives us a strictly positive constant F, independent of i, such that, for $n > N_1(0)$,

$$\frac{\mu\left(J_n(0) \bigcap \{y : | f_{q_n^k}(y) | \in \mathcal{K}\}\right)}{\mu(Q_n)} > F \frac{H(0)}{4}.$$

So, by monotonicity, for $n > N_1(0)$ there exists a strictly positive constant W, independent of i, such that

$$\frac{\mu\left(Q_n \bigcap \{y : | f_{q_n^k}(y) | \in \mathcal{K}\}\right)}{\mu(Q_n)} > W. \tag{11}$$

This proves our claim.

Now from Lemma 2.2.11, given $0 < \epsilon' < \frac{1}{2}\mu(B)$, there is a $B_0 \subseteq B$ with $\mu(B \setminus B_0) < \epsilon'$, and a $\delta > 0$, such that, for all $x \in B_0$,

$$\frac{\mu\left(I'(x) \bigcap B\right)}{\mu\left(I'(x)\right)} > 1 - \frac{1}{2} W \tag{12}$$

for any interval $I'(x)$ containing x, of length $\mu\left(I'(x)\right) < \delta$. Then, from Lemma 2.2.10, given $0 < \epsilon'' < \frac{1}{2}\mu(B_0)$, there is a Borel set $B_1 \subseteq B_0$ with $\mu(B_0 \setminus B_1) < \epsilon''$, and a sequence $\{n_j\}_{j \geq 1}$ of integers, with $T^{q_{n_j}^k} x \in B_0$ for all $x \in B_1$. We now fix $x \in B_1$. From Proposition 2.2.2 we know that, given $\delta > 0$, there is an integer M' such that, for $k > M'$,

$$\mu\left(Q_{i_k}^k(x)\right) < \delta,$$

where $Q^k_{i_k}(x)$ is the interval in $P_k(\alpha)$ which contains x. So for $n = n_j > M'$, we have that

$$\frac{\mu(Q_n(x) \bigcap B)}{\mu(Q_n(x))} > 1 - \frac{1}{2}W. \tag{13}$$

Now, applying (12) to $T^{q_n^k}x \in B_0$ for this n, we also get that

$$\frac{\mu\left(T^{q_n^k}Q_n(x) \bigcap B\right)}{\mu\left(T^{q_n^k}Q_n(x)\right)} > 1 - \frac{1}{2}W, \tag{14}$$

since the interval $T^{q_n^k}Q_n(x)$ encloses the point $T^{q_n^k}x$ and

$$\mu\left(T^{q_n^k}Q_n(x)\right) = \mu(Q_n(x)) < \delta.$$

So, using the T-invariance of μ, (14) becomes

$$\frac{\mu\left(Q_n(x) \bigcap T^{-q_n^k}B\right)}{\mu(Q_n(x))} > 1 - \frac{1}{2}W. \tag{15}$$

By (13) and (15) B and $T^{-q_n^k}B$ take up, in proportion, more than $1 - \frac{1}{2}W$ of $Q_n(x)$ for $n = n_j > M'$. Hence we have that for $n = n_j > M'$

$$\frac{\mu\left(Q_n(x) \bigcap B \bigcap T^{-q_n^k}B\right)}{\mu(Q_n(x))} > 1 - W.$$

So, reducing to a subset of B if necessary, we find that, for our chosen n and x, the set $B \bigcap T^{-q_n^k}B$ takes up at least a fixed proportion of $Q_n(x)$; from (11) we have that for n sufficiently large $\{y : | f_{q_n^k}(y) | \in \mathcal{K}\}$ also takes up at least a fixed proportion of $Q_n(x)$. Since these proportions add up to more than 1, the two sets must intersect for $n = n_j > Y' = \max\{M', N_1(0)\}$.

This shows that there is an m with

$$B \bigcap T^{-m}B \bigcap \{y : | f_m(y) | \in \mathcal{K}\} \neq \emptyset.$$

Thus we have a contradiction, and the lemma is proved. ∎

2.2.13 THEOREM: *Let $\int f d\mu = 0$, $f^{(k-2)}$ be continuous with zero integral, $f^{(k-1)}$ be piecewise continuous also with zero integral and $f^{(k)}$ Riemann integrable with $\int f^{(k)} d\mu \neq 0$. Then the skew-product T_f is ergodic for all $\alpha \in \mathcal{S}_k$.*

Proof: Since the only closed additive subgroups of $\mathbf{R}$ are: $\lambda\mathbf{Z}$ for $\lambda \geq 0$, or $\mathbf{R}$ itself, then the above Lemma and 1.2.2 demonstrate that the only possible remaining choice for the essential values is $E(f) = \mathbf{R}$. By 1.2.5 this shows that the skew-product T_f is ergodic. ∎

2.2.14 Remarks: We note that for $k = 1, \mathcal{S}_1 \neq \mathbf{R} \setminus \mathbf{Q}$, however the result from [P1] is valid for all irrationals. This is due to us not having to use the 'inductive step' 2.2.9 for $k = 1$. For $k > 1$ this step introduces the extra power of k necessary to guarantee the result. Also for $k > 1$ we have that $\mathcal{S}_k$ is a dense G_δ of measure zero (see [HW]). ∎

3. The Adding Machine

3.1 INTRODUCTION. In this section we study skew product extensions of the adding machine transformation on $\mathbf{T}$. Firstly we study the class of functions which are piecewise continuous, have zero integral and have a derivative which is Riemann integrable with non-zero integral. We study this class of functions and show that their properties ensure that the skew-product is ergodic. Then we note that the method used in Section 2 will not work for the adding machine in the case where $k > 1$.

Finally we study the class of functions with zero integral and have a derivative of bounded variation; we show that these properties guarantee that these functions define cocycles which are always coboundaries for a certain class of adding machine.

3.2 CONDITIONS FOR ERGODICITY.

3.2.1 Convention: Throughout this section we shall suppose that f is piecewise continuous with zero integral, and f' is Riemann integrable with non-zero integral.

Let f have discontinuities at $0 \leq \omega_1 < \cdots < \omega_N < 1$, as in Section 2, and let $d_r = f(\omega_r^+) - f(\omega_r^-)$ for $r = 1, \ldots, N$, where $f(\omega_r^+)$ and $f(\omega_r^-)$ are the limits of f at ω_r as x approaches from less or greater argument. We suppose without loss of generality that $\Sigma_{r=1}^{N} d_r < 0$.

We define P_n to be the partition of $\mathbf{T}$ into $p(n)$ intervals (n-cylinders), each of length $1/p(n)$, defined by the first $p(n)-1$ points on the orbit of 0 under T. The subscript of a cylinder denotes the P_n to which it belongs.

3.2.2 LEMMA: *For all n and all cylinders $Q_n \in P_n$, there is a sub-interval (union of sub-cylinders) $J_n \subset Q_n$ on which $f_{p(n)}$ is continuous, satisfying*

$$\frac{\mu(J_n)}{\mu(Q_n)} \geq \frac{1}{N}.$$

Proof: Clearly, for all n the discontinuities of $f_{p(n)}$ occur at the points $T^s\omega_r$ for $s = 0, \ldots, p(n)-1$ and $r = 1, \ldots, N$. In any cylinder $Q_n \in P_n$ there are, therefore, at most N discontinuities of $f_{p(n)}$, and hence the result follows. ∎

The following may be found in [P2, p.32]:

3.2.3 LEMMA: *There are positive constants K_1, K_2 and an integer N_1 such that for $n \geq N_1$ we have that for μ-a.e. $x \in \mathbf{T}$, $n.K_1 < f_n'(x) < n.K_2$.*

3.2.4 LEMMA: *There exist strictly positive constants H, F and an integer N_1 with the following properties: Within each cylinder Q_n of P_n for $i = 1, \ldots, p(n)$, there is a sub-interval J_n such that for $n > N_1$,*
(i) $f_{p(n)}$ moves through a height greater than H on J_n,
(ii) for any interval $[x, y] = I_n \subseteq f_{p(n)}(J_n)$, we have that

$$\frac{\mu\left(f_{p(n)}^{-1}(I_n) \bigcap J_n\right)}{\mu(Q_n)} > F \mid y - x \mid .$$

Proof: By Lemma 3.2.2 we may consider the cylinder $J_n = [a_n, b_n) \subset Q_n$ on which $f_{p(n)}$ is continuous and which satisfies

$$p(n) \,.\, \mu(J_n) = \frac{\mu(J_n)}{\mu(Q_n)} \geq \frac{1}{N}. \tag{16}$$

Then from Lemma 3.2.3, since $p(n) > n$ for all n, we have, for $n > N_1$, that

$$K_1.p(n) < f_{p(n)}'(x) < K_2.p(n), \tag{17}$$

for μ-a.e. $x \in \mathbf{T}$. Now we choose $H = K_1/2N$, then for $n > N_1$ the height h moved through by the cocycle on J_n is given by

$$h = f_{p(n)}(b_n) - f_{p(n)}(a_n) = \int_{a_n}^{b_n} f_{p(n)}' d\mu.$$

In order to obtain a lower bound for this height, we use (16) and (17) above to give us $h > K_1 . \mu(J_n) . p(n) \geq \frac{K_1}{N}$. This proves (i).

For (ii) define $m = f_{p(n)}^{-1}(x) \bigcap J_n$ and $p = f_{p(n)}^{-1}(y) \bigcap J_n$; then, since $f_{p(n)}$ is strictly increasing for $n > N_1$, we have that $f_{p(n)}^{-1}(I_n) \bigcap J_n = [m, p)$, and

$$y - x = \int_m^p f'_{p(n)} d\mu.$$

So, from (17) above we have, for $n > N_1$, that

$$p - m > \frac{y - x}{K_2 . p(n)}.$$

Hence we have that

$$\frac{p - m}{\mu(Q_n)} > \frac{y - x}{K_2},$$

and so

$$\frac{\mu\left(f_{p(n)}^{-1}(I_n) \bigcap J_n\right)}{\mu(Q_n)} > \frac{y - x}{K_2}$$

for $n > N_1$. Putting $F = 1/K_2$, we note that F is strictly positive and independent of i, which completes the proof of the second assertion. ∎

3.2.5 LEMMA: *$E(f) \neq \lambda \mathbf{Z}$ for any $\lambda \geq 0$.*

Proof: Suppose that $E(f) = \lambda \mathbf{Z}$ where $\lambda \geq 0$. Since, by hypothesis, f has bounded variation, applying the Denjoy–Koksma inequality gives us a $c > 0$ such that $| f_{p(n)}(x) | \leq c$ for all n, and all $x \in \mathbf{T}$. Let v be the greatest integer such that $v\lambda < c$, with $v = 0$ if $\lambda = 0$. We choose

$$0 < \epsilon < \frac{H}{4(2v + 1)}$$

where H is the number obtained in Lemma 3.2.4(i). As in [P1, Lemma 2.10] we consider the compact set

$$\mathcal{K} = (-\epsilon, c] \setminus \bigcup_{i=0}^{v} (i\lambda - \epsilon, i\lambda + \epsilon).$$

Clearly $E(f) \bigcap \mathcal{K} = \emptyset$, so applying Lemma 1.2.3 we obtain a Borel set such that $B \bigcap T^{-m} B \bigcap \{y : | f_m(y) | \in \mathcal{K}\} = \emptyset$ for all $m \in \mathbf{Z}$.

As in 2.2.12, using 3.2.4 we may show that for all $n > N_1$, and every cylinder $Q_n \in P_n$, there is a strictly positive constant W, independent of i, with

$$\frac{\mu\left(Q_n \bigcap \{y :| f_{p(n)}(y) | \in \mathcal{K}\}\right)}{\mu(Q_n)} > W. \tag{18}$$

Now, from the Lebesgue Density Theorem, given $x \in B \in \Omega$ and an $\epsilon = \frac{1}{2}W > 0$, there is an N_2 such that for $n > N_2$

$$\frac{\mu\left(Q_n(x) \bigcap B\right)}{\mu\left(Q_n(x)\right)} > 1 - \frac{1}{2}W,$$

where $x \in Q_n(x) \in P_n$. Also, since $T^{p(n)}Q_n(x) = Q_n(x)$ and the measure μ is T-invariant, we have that

$$\frac{\mu\left(Q_n(x) \bigcap T^{-p(n)}B \bigcap B\right)}{\mu\left(Q_n(x)\right)} > 1 - W.$$

So, for $n > \max\{N_1, N_2\}$, we have that the set $B \bigcap T^{-p(n)}B$ takes up at least a fixed proportion of the cylinder $Q_n(x)$; from (18) above we also have that $\{y :| f_{p(n)}(y) | \in \mathcal{K}\}$ takes up at least a fixed proportion of $Q_n(x)$, for all i. Since these proportions add up to more than 1, the two sets must intersect for this n.

This shows that there is an m with $B \bigcap T^{-m}B \bigcap \{y :| f_m(y) | \in \mathcal{K}\} \neq \emptyset$. Thus we have a contradiction, and the lemma is proved. ∎

Thus we may easily show:

3.2.6 THEOREM: *Let f be piecewise continuous, with $\int f d\mu = 0$, f' Riemann integrable and $\int f' d\mu \neq 0$. Then the skew product T_f is ergodic.*

3.2.7 Remark: We note that the arguments above and in Section 2 involve properties of the sequences of integers $p(n)$, q_n respectively associated to the map T. It seems that we may obtain similar results to those given in Section 2 for the adding machine transformation. However, in Section 2, for the class of functions defined for $k > 1$ in 2.2.1, we use the 'good approximation' conditions of our irrational numbers to ensure that, as we iterate our partition intervals by T^{q_n}, the discontinuities do not spread evenly over the interval. For the adding machine, this behaviour does not occur, and so the method will not be applicable. ∎

3.3. COBOUNDARIES FOR THE ADDING MACHINE.

3.3.1 CONVENTION: *In this subsection we study the class of functions with zero integral and have a derivative which has bounded variation. We also assume that the associated integers $p(n)$ (see 1.4) of our adding machine transformation satisfy*

$$\sum_{n=1}^{\infty} \frac{q_{n+1}}{p(n)} < \infty. \tag{19}$$

3.3.2 LEMMA: *Suppose f is as above, then there is a positive constant K such that, for all n and all $x \in \mathbf{T}$,*

$$| f_{p(n)}(x) | < \frac{K}{p(n)}. \tag{20}$$

Proof: Since f' has bounded variation and $\int f' d\mu = 0$, the Denjoy–Koksma inequality gives us that

$$| f'_{p(n)}(x) | \leq c', \tag{21}$$

for some positive constant $c' = \operatorname{var} f'$, all $x \in \mathbf{T}$ and all n. Since $\int f d\mu = 0$, we also note that for all n, we have that

$$\int_{B_n} f_{p(n)} d\mu = 0, \tag{22}$$

on any n-cylinder, $B_n \subset \mathbf{T}$. By hypothesis f is continuous, so we may assume that there is a positive constant c such that $| f(x) | < c$ for all $x \in \mathbf{T}$.

Let $K = \max\{c, 3c'\}$, then suppose, without loss of generality, that there is an $x \in \mathbf{T}$ and an n such that $f_{p(n)}(x) \geq K/p(n)$. We show that if this is so, then there is an n-cylinder on which the integral of $f_{p(n)}$ cannot be zero.

From (21) the slope of $f_{p(n)}$ is bounded for all n, hence we may calculate the proportion of the n-cylinder to which x belongs on which $f_{p(n)}$ is positive. Let d_1, d_2 denote the distances to the right and left of x respectively, when $f_{p(n)}$ next crosses the x-axis. Since f is absolutely continuous we have that

$$f_{p(n)}(x) = \int_x^{x+d_1} f'_{p(n)} d\mu,$$

and so, by assumption and (21),

$$d_1 > \frac{K}{c'} . \frac{1}{p(n)} \geq 3 . \frac{1}{p(n)}.$$

We get a similar inequality for d_2. This implies that the function $f_{p(n)}$ cannot return to zero inside the cylinder. Therefore we cannot have $\int f_{p(n)} d\mu = 0$ for this cylinder; this contradicts (22), and so the result follows. ∎

3.3.3 THEOREM: *Suppose that $f : \mathbf{T} \to \mathbf{R}$ is such that $\int f d\mu = 0$, with f' of bounded variation, then for all $x \in \mathbf{T}$ we have that $\sup_n \mid f_n(x) \mid < \infty$, and so f defines a L^∞ coboundary.*

Proof: Any $n \in \mathbf{N}$ may be written uniquely as

$$n = \sum_{i=0}^{s(n)} n_i p(i) \quad \text{where } n_i \in \{0, \ldots, q_{i+1} - 1\},$$

for some positive integer $s(n)$. Thus for all $x \in \mathbf{T}$, we may write $f_n(x)$ as

$$f_n(x) = f_{p(0)n_0}(x) + f_{p(1)n_1}(T^{n_0}x) + \cdots + f_{p(s(n))n_{s(n)}}(T^{\sum_{i=0}^{s(n)-1} p(i)n_i}x),$$

by the cocycle relation. Hence by the cocycle relation (3) again, and the triangle inequality, we have that

$$\mid f_n(x) \mid \leq |n_0 f(x)| + \cdots + \mid n_{s(n)} f_{p(s(n))}(T^{\sum_{i=0}^{s(n)-1} p(i)n_i}x) \mid .$$

Using (20) we have that

$$\mid f_n(x) \mid \leq n_0.K + \cdots + n_{s(n)}.\frac{K}{p(s(n))} < K.\sum_{i=0}^{s(n)} \frac{q_{i+1}}{p(i)} < \infty,$$

since $n_i < q_{i+1}$, and using our hypothesis (19). Hence $\sup_n \mid f_n(x) \mid < \infty$ for all $x \in \mathbf{T}$, and the result follows from [AS]. ∎

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